The University of Alabama

College of Engineering · Computer Science

Assignment 1

Submission Instructions

- Submit your solutions as a **separate PDF file**.
- Clearly label each problem number and part (e.g., Problem 1a, Problem 2c).
- For Python problems, include screenshots of your code and output.
- Ensure your work is legible and well-organized. Show all steps for full credit.

Problem 1: Vector Spaces and Binary Arithmetic (15 Points)

Consider the vector space $(\mathbb{Z}_2^3, \mathbb{Z}_2, +, \cdot)$ where $\mathbb{Z}_2 = \{0, 1\}$ with arithmetic modulo 2.

(a) Compute $\boldsymbol{u} + \boldsymbol{v}$ for $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (1, 1, 1)$ in $\mathbb{Z}_2^3$.

(b) Prove that in $\mathbb{Z}_2^n$, every vector is its own additive inverse. That is, show $\boldsymbol{u} + \boldsymbol{u} = \mathbf{0}$ for any $\boldsymbol{u} \in \mathbb{Z}_2^n$.

(c) Is the set $S = \{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$ a basis for $\mathbb{Z}_2^3$? Justify your answer.

Solution:

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Problem 2: Linear Independence and Basis (15 Points)

Consider the following sets of vectors in $\mathbb{R}^3$:

(a) Determine whether $S_1 = \{(1,0,0),(1,1,0),(1,1,1)\}$ is linearly independent. If so, is it a basis for $\mathbb{R}^3$?

(b) Determine whether $S_2 = \{(1,2,3),(2,4,6),(0,1,1)\}$ is linearly independent. If not, identify the dependency.

(c) Suppose you have two vectors in $\mathbb{R}^2$ and their span equals $\mathbb{R}^2$. What can you conclude about their linear independence?

Solution:

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Problem 3: Change of Basis (10 Points)

Let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ where $\boldsymbol{b}_1 = (1, 1)$ and $\boldsymbol{b}_2 = (1, -1)$ be a basis for $\mathbb{R}^2$.

(a) Express the vector $\boldsymbol{v} = (5, 1)$ as a linear combination of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. What are the coordinates $[\boldsymbol{v}]_{\mathcal{B}}$?

(b) Express the vector $\boldsymbol{v} = (1, 5)$ as a linear combination of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. What are the coordinates $[\boldsymbol{v}]_{\mathcal{B}}$?

(c) Find the matrix P (the inverse of the basis matrix) that converts standard coordinates to $\mathcal{B}$-coordinates.

(d) Verify your answer to (a) by computing $P \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and checking if it matches your result from part (a).

Solution:

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Problem 4: Vector Norms and Their Properties (10 Points)

Let $\boldsymbol{x} = \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix}$.

(a) Compute $\|\boldsymbol{x}\|_1$, $\|\boldsymbol{x}\|_2$, and $\|\boldsymbol{x}\|_\infty$.

(b) Normalize $\boldsymbol{x}$ to obtain the unit vector $\hat{\boldsymbol{x}}$ (using the ℓ_2 norm).

Solution:

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Problem 5: Inner Products, Angles, and Projections (20 Points)

Consider the vectors $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

(a) Compute the dot product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$.

(b) Compute the cosine of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Are the vectors acute, obtuse, or orthogonal?

(c) Compute the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ (i.e., $\text{comp}_{\boldsymbol{a}}(\boldsymbol{b})$).

(d) Compute the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ (i.e., $\text{proj}_{\boldsymbol{a}}(\boldsymbol{b})$).

Solution:

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Problem 6: Gram-Schmidt Orthogonalization (15 Points)

Apply the Gram-Schmidt process to convert the following linearly independent set into an orthonormal basis for $\mathbb{R}^2$:

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(a) Find the orthogonal vector $\boldsymbol{v}_2 = \boldsymbol{u}_2 - \text{proj}_{\boldsymbol{u}_1}(\boldsymbol{u}_2)$.

(b) Normalize both $\boldsymbol{u}_1$ and $\boldsymbol{v}_2$ to obtain the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

(c) Verify that $\langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle = 0$ and $\|\boldsymbol{e}_1\|_2 = \|\boldsymbol{e}_2\|_2 = 1$.

Solution:

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Problem 7: Python: Vectorized Computations (15 Points)

Complete the following Python functions using NumPy to perform the following tasks. You can append images of your code to the solution section. **Do not use explicit for-loops**; use vectorized operations only.

(a) Write a function `compute_norms(x)` that takes a vector `x` and returns a tuple of $(\|\boldsymbol{x}\|_1, \|\boldsymbol{x}\|_2, \|\boldsymbol{x}\|_\infty)$.

(b) Write a function `cosine_similarity(a, b)` that computes the cosine similarity between two vectors.

(c) Write a function `project(b, a)` that computes the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

Test each function with the vectors from Problems 4 and 5.

Solution:

```python
import numpy as np

# (a) Compute all three norms
def compute_norms(x):
    # Your code here










# (b) Cosine similarity
def cosine_similarity(a, b):
    # Your code here






# (c) Vector projection
def project(b, a):
    # Your code here






# Test your functions here
```

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